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A NOTE ON A BIFURCATION PROBLEM IN FINITE PLASTICITY RELATED TO VOID NUCLEATION

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Abstract—In this paper, a bifurcation problem for a solid sphere subjected to a monotonically increasing, radial, tensile, dead load p at its outer boundary is examined. The material is assumed to obey a finite strain version of J_2-flow theory. One solution to this problem, for all values of p, corresponds to a homogeneous state. However, for a certain critical range of p, there is in addition, a second possible configuration, this one involving an internal spherical cavity. The classical infinitesimal strain theory of plasticity does not exhibit such a bifurcation.

1. INTRODUCTION

In a recent paper[1], a class of bifurcation problems for the equations of non-linear elasticity were examined by Ball. These bifurcation problems are concerned with the phenomenon of internal rupture, in which a hole forms in the interior of a solid body which contains no hole in the undeformed state. An alternative physical interpretation for such problems in terms of the growth of a pre-existing micro-void is given in Ref. [2]. The purpose of the present note is to analyze a corresponding bifurcation problem within the context of plasticity theory.

We consider an incompressible solid sphere under symmetric, monotonic increasing, tensile dead load p. The constitutive relation describing the material behavior is taken to be a generalization of J_2-flow to finite deformations. One solution to this problem, for all values of p, corresponds to a homogeneous state in which the sphere remains undeformed but stressed. However, for a certain critical range of p, one has in addition, a second possible configuration involving an internal spherical cavity. An explicit expression for the critical load p_{cr} at which the cavity is initiated is obtained (see eqn (14)). It is important to note that this critical load is given automatically by the analysis and does not involve any *ad hoc* assumptions. The relation between applied load and cavity radius for subsequent cavity growth is also established (see eqn (13)).

It is worth pointing out that the bifurcation considered here is inherently associated with the kinematic nonlinearity. When the present problem is examined using classical infinitesimal strain plasticity theory, one finds through a formal calculation (see Ref. [3]) that p_{cr} is given by

$$p_{cr} = \frac{2}{3}\int_0^\infty \frac{\hat{\sigma}(\varepsilon)}{\varepsilon}\,d\varepsilon$$

where $\sigma = \hat{\sigma}(\varepsilon)$ describes the stress–strain relation of the material in monotonic uni-axial tension. Under usual conditions, $[\hat{\sigma}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$, $\hat{\sigma}(\varepsilon) = O(\varepsilon^n)$ as $\varepsilon \to \infty$, $n \geqslant 0]$, the above integral is clearly unbounded and so bifurcation is *not* predicted by the infinitesimal theory at any finite load.

2. FORMULATION AND SOLUTION

Consider a solid sphere of radius A, subjected to a monotonically increasing radial tension (dead load), $p(t)$, applied to its surface $R = A$. In view of symmetry, the resulting deformation of the sphere is described by

$$r = r(R,t), \qquad \theta = \Theta \quad \text{and} \quad \phi = \Phi, \qquad r(0+,t) \geqslant 0 \tag{1}$$

where (r, θ, ϕ) are the current spherical polar coordinates of the point which, in the undeformed configuration, was located at (R, Θ, Φ). If the material is assumed to be incompressible, the deformation gradient $\mathbf{F}$ obeys $\det \mathbf{F} = 1$. For deformation (1), this implies $r^2\,\partial r/\partial R = R^2$, which when integrated gives

$$r = r(R,t) = \{R^3 + c^3(t)\}^{1/3}, \qquad c(t) \geqslant 0 \tag{2}$$

where $c(t)$ is to be determined. If it is found that $c(t) = 0$, eqn (2) implies that the body remains a solid sphere in the current configuration. On the other hand, if $c(t)$ is found to be positive (i.e. $r(0+,t) > 0$), there is a cavity of radius c centered at the origin in the current configuration.

From eqns (1) and (2), the non-vanishing components of the Eulerian strain-rate tensor are found to be

$$D_r = -2\dot{r}/r, \qquad D_\theta = D_\phi = \dot{r}/r \tag{3}$$

where the dot denotes the Lagrangian time derivative. In view of symmetry, and assuming the material to be isotropic, the non-zero components of the (Cauchy) true stress tensor are the radial stress $\sigma_r(r,t)$ and the hoop stresses $\sigma_\theta(r,t) = \sigma_\phi(r,t)$. The prescribed dead load boundary condition on the surface of the sphere requires that

$$\sigma_r(a,t) = p(t)\,(A/a)^2 \tag{4}$$

where $a = r(A,t) = \{A^3 + c^3\}^{1/3}$ represents the deformed outer radius.

The constitutive relation for the elastic–plastic material is taken to be a generalization of J_2-flow theory to finite deformations (see e.g. Ref. [4])

$$\mathbf{D} = (3E/2)\overset{\nabla}{\mathbf{S}} + \Lambda(3\sigma_e/2)\dot{\varepsilon}_p(\sigma_e)\mathbf{S}. \tag{5}$$

Here $\mathbf{S}$ is the deviatoric Cauchy stress, σ_e is the effective Cauchy stress ; Λ is a loading coefficient ; $\varepsilon_p(\cdot)$ is a given constitutive function representing the effective plastic logarithmic strain. The Jaumann (co-rotational) rate of the Cauchy stress deviator is denoted by $\overset{\nabla}{\mathbf{S}}$, so that $\overset{\nabla}{\mathbf{S}} = \dot{\mathbf{S}} - \mathbf{\Omega}\mathbf{S} + \mathbf{S}\mathbf{\Omega}$ where $\mathbf{\Omega}$ is the spin-tensor. In the case of uni-axial tension, the relation between true stress σ and the logarithmic strain ε, in monotonic loading, can be obtained from eqn (5) as $\varepsilon = \hat{\varepsilon}(\sigma) \equiv \sigma/E + \varepsilon_p(\sigma)$. We assume this relation to be invertible so that we may write the *stress–strain relation in uni-axial tension* as either

$$\sigma = \hat{\sigma}(\varepsilon) \quad \text{or} \quad \varepsilon = \hat{\varepsilon}(\sigma). \tag{6}$$

In the present problem, $\mathbf{\Omega}$ vanishes, and thus $\overset{\nabla}{\mathbf{S}} = \dot{\mathbf{S}}$. Also, $\sigma_e = \sigma_\theta - \sigma_r$. Equations (5) and (6), under conditions of loading ($\Lambda = 1$), yield

$$\dot{\hat{\varepsilon}}(\sigma_e) = 2\dot{r}/r, \qquad \sigma_e = \sigma_\theta - \sigma_r. \tag{7}$$

On using eqns (1), we may integrate eqns (7) with respect to the parameter t to obtain $\hat{\varepsilon}(\sigma_e) = 2\ln(r/R)$. Using eqns (6) to invert this gives

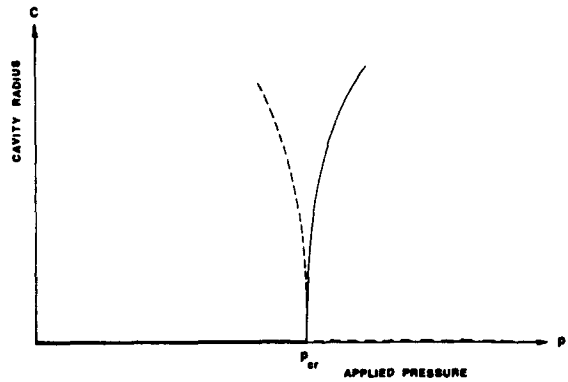

Fig. 1. Schematic graph showing variation of cavity radius c vs applied load p.

$$\sigma_c = \hat{\sigma}\{2 \ln (r/R)\}. \tag{8}$$

Finally, in the absence of body forces, the equilibrium equations reduce to the single equation

$$\frac{\partial \sigma_r}{\partial r} - 2\frac{\sigma_c}{r} = 0. \tag{9}$$

Thus, *the problem to be solved is the following* : we wish to find† $\sigma_r(r, t)$ and $c(t) \geqslant 0$ such that the field equations, eqns (2), (8), and (9), and boundary condition (4) hold. In addition, if $c(t) > 0$ it is also required that

$$\sigma_r(c, t) = 0. \tag{10}$$

This stipulates that when a hole appears at the origin, it must be traction free.

First, it is readily shown that, for all values of $p \geqslant 0$, one solution to the foregoing problem is

$$\sigma_r(r, t) = p(t), \qquad c(t) = 0. \tag{11}$$

This corresponds to a homogeneous state of deformation $r = \hat{r}(R, t) = R$, with resulting stresses $\sigma_r = \sigma_\theta = \sigma_\phi = p(t)$.

Next we seek a solution for which $c(t) > 0$. Combining eqns (2), (8) and (9), integrating with respect to r, using boundary condition (4), and employing a change of variables yields

$$\sigma_r(r, t) = p\left(\frac{A}{a}\right)^2 - \int_{2\ln(a/A)}^{2\ln(r/R)} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2) - 1}\, d\varepsilon, \qquad R = (r^3 - c^3)^{1/3}. \tag{12}$$

On enforcing the remaining boundary condition (10), one is led to

$$p = \left(\frac{a}{A}\right)^2 \int_{2\ln(a/A)}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2) - 1}\, d\varepsilon, \qquad a = (A^3 + c^3)^{1/3}. \tag{13}$$

Thus, if for a given value of p, eqn (13) can be solved for a positive root c, then this c, together with eqn (12), provides a solution to the problem at hand.

It is readily shown that under the usual constitutive conditions, $[\hat{\sigma}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$, $\sigma(\varepsilon) = O(\varepsilon^n)$ as $\varepsilon \to \infty$, $n \geqslant 0]$, the integral in eqn (13) is bounded for all $a \geqslant A$. Therefore, there exists a value of pressure $p(>0)$ corresponding to each $c > 0$. A schematic graph of p vs c is shown in Fig. 1. *The critical load* p_{cr} at which the cavity is initiated is found by letting $c \to 0+$ in eqn (12), i.e.

† The remaining physical quantities can be immediately found thereafter from eqns (2), (9) and $\sigma_\theta = \sigma_\phi = \sigma_c + \sigma_r$.

$$p_{cr} = \int_0^\infty \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2)-1}\,d\varepsilon. \tag{14}†$$

As noted previously, the integral in eqn (14) is *bounded* and so, the cavity is initiated at a *finite* value of load.

3. DISCUSSION

For all values of the prescribed radial dead load p one possible configuration is that in which the sphere remains solid (see eqns (11)). On the other hand, for a certain range of p one has, in addition, a second possible configuration involving an internal spherical cavity. Equation (14) gives the critical value of the load, p_{cr}, at which a cavity may initiate.

It is necessary to examine the stability of these two possible configurations in order to determine whether the homogeneous solution will in fact bifurcate, at $p = p_{cr}$, into the one involving a cavity.

To carry out this stability analysis, we make use of an energy criterion developed by Hill[5] and Petryk[6] for quasi-static deformations of general elastic–plastic solids occupying the domain V and subject to dead load tractions on its boundary S. Thus we consider the energy functional

$$E(\mathbf{v}, t) = \int_0^t \left\{ \int_V s_{ij} v_{j,i}\,dV - \int_S t_i v_i\,dS \right\} d\tau \tag{15}$$

defined for all kinematically admissible velocity fields $\mathbf{v}$, where s_{ij} denote the components of the nominal stress tensor and t_i the components of the nominal traction vector. A Taylor expansion in eqn (15) yields

$$E(\mathbf{v}, t+\delta t) = E(t) + E_1\delta t + E_2(\delta t)^2 + o(\delta t)^2 \quad \text{as} \quad \delta t \to 0 \tag{16}$$

where

$$E_1 = \frac{dE}{dt} \tag{17}$$

$$E_2 = \frac{1}{2}\frac{d^2E}{dt^2} = \frac{1}{2}\frac{dE_1}{dt}. \tag{18}$$

The equilibrium configuration is found by setting

$$E_1 = 0 \tag{19}$$

and this configuration is *stable*[5, 6] if

$$E_2 > 0 \quad \text{when} \quad E_1 = 0. \tag{20}$$

For the problem of concern in this paper, we see that

† Note that the formula for p_{cr} according to the small strain theory (see Introduction) may be obtained formally by replacing the exponential in eqn (14) by the first two terms in its Taylor expansion about $\varepsilon = 0$.

$$E_1 = \int_V \sigma_{ij} D_{ij} \, dV - \int_S t_i v_i \, dS \tag{21}$$

$$= \int_V \sigma_e \dot{\varepsilon} \, dV - \int_S t_i v_i \, dS \tag{22}$$

where σ_e and $\dot{\varepsilon}$ are given by eqns (8) and (7), respectively. Thus we find that

$$E_1 = 4\pi c^2 \dot{c} \left[\int_{2\ln(a/A)}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2) - 1} \, d\varepsilon - p \left(\frac{A}{a} \right)^2 \right] \tag{23}$$

and so by eqn (19) there are two equilibrium configurations, namely that corresponding to $c = 0$ (homogeneous solution) and that corresponding to

$$p = \hat{p}(c) \equiv \left(\frac{a}{A} \right)^2 \int_{2\ln(a/A)}^{\infty} \frac{\hat{\sigma}(\varepsilon)}{\exp(3\varepsilon/2) - 1} \, d\varepsilon, \qquad a = (A^3 + c^3)^{1/3}. \tag{24}$$

These are, of course, the same equilibrium configurations found previously.

It is readily verified that, for the homogeneous solution, $E_2 = 0$. However, for the bifurcated solution, it can be shown on using eqns (23) and (24) that

$$E_2 = \frac{2\pi c^2 \dot{c}^2}{(1 + c^3/A^3)^{2/3}} \hat{p}'(c) \tag{25}$$

and so the bifurcated solution is stable provided $\hat{p}'(c) > 0$.

Figure 1 shows schematically, a graph of the cavity radius c vs the applied load p. The bold horizontal line coinciding with the positive p-axis corresponds to the homogeneous solution. The curves emanating from $(p_{cr}, 0)$ correspond to a bifurcated solution involving a cavity. If such a curve comes off to the right, the bifurcated solution is locally stable and so the sphere would indeed develop an internal cavity at $p = p_{cr}$. Conversely, if bifurcation to the left occurs, the solution is locally unstable and the sphere remains solid.

On using a Taylor expansion near p_{cr}, eqn (13) yields

$$p = p_{cr} + \frac{2}{3} \frac{c^3}{A^3} \left(p_{cr} - \frac{2E}{3} \right) + o(c^3) \quad \text{as} \quad c \to 0 \tag{26}$$

where $E = \hat{\sigma}'(0)$ is Young's modulus. Thus when $p_{cr} > 2E/3$, the slope of the curve at $(p_{cr}, 0)$ is positive and so bifurcation to the right occurs. On the other hand, when $p_{cr} < 2E/3$, the slope is negative. Consequently a void will actually appear at $p = p_{cr}$ only if $p_{cr} > 2E/3$.

Of course, the load level at which stable bifurcation is predicted here is unreasonably large. This feature is commonly encountered in bifurcation analyses employing classical flow theories of plasticity. It may be possible to use more elaborate constitutive models or to include the effect of a pre-existing stress concentrator (such as an inclusion) in order to obtain more realistic values for the critical load.

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